2.5 - The second derivative

- The derivative of the derivative
- The derivative of $f'(x)$

**NOTATION**

$f''(x)$, $y''$ ("double prime")

$\frac{d^2 y}{dx^2}$

**Recall**: if $f'(x) > 0$, $f$ is INC
if $f'(x) < 0$, $f$ is DEC
if $f'(x) = 0$, $f$ is constant
So... for all \( x \in (a, b) \) \((a < x < b)\)

\( \text{If } f''(x) > 0, \ f'(x) \text{ is INC } \) (slope is increasing)

So \( f \) is curving upward or

\( f \) is concave up

\( \text{If } f''(x) < 0, \ f'(x) \text{ is DEC, so} \)

\( f \) is concave down

\( \text{If } f''(x) = 0, \ f'(x) \text{ is constant} \)

(no curvature/concavity)

So \( f \) is linear on \((a, b)\)
The graph of $f(x)$ is shown in Figure 2.18. Which of the following are true for $f$ as shown in this window?

(a) $f(x)$ is positive  
(b) $f(x)$ is increasing  
(c) $f'(x)$ is positive  
(d) $f'(x)$ is increasing  
(e) $f''(x)$ is non-negative

4 possibilities on any interval (ignoring lines)

1. $f'(x) < 0$  
   $f''(x) > 0$

2. $f'(x) > 0$  
   $f''(x) < 0$

3. $f'(x) < 0$  
   $f''(x) < 0$

4. $f'(x) > 0$  
   $f''(x) > 0$
Two cars:

- Distance (feet): 100
- Time (sec)

This car is accelerating.

- Distance (ft)
- Time (sec)

This car is decelerating.
Sketch a graph of the second derivative of the following:
In words: a function is differentiable at a point if the slope as you approach from the left is equal to the slope as you approach from the right.

Symbolically:

$$\lim_{x \to c^-} f'(x) = \lim_{x \to c^+} f'(x)$$

* If \( f \) is differentiable at all \( c \in (a, b) \) then \( f \) is differentiable on \( (a, b) \).

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Look to be non-differentiable at \((0,0)\)

Look to be differentiable.
To be differentiable at a point:

1. must be continuous
2. no sharp corners
3. no vertical tangent lines

\[ y = \sqrt[3]{x} \]

not differentiable at \((0,0)\) vertical tangent line
Is the following differentiable at $x=1$?

\[ g(x) = \begin{cases} x, & x < 1 \\ x^6, & x \geq 1 \end{cases} \]

**$g$ is continuous at $x=1$:**

\[ \lim_{x \to 1^-} g(x) = \lim_{x \to 1^+} g(x) \]

\[ \lim_{x \to 1^-} x = \lim_{x \to 1^+} x^6 \]

\[ 1 = 1 \]

\[ 1 = 1 \quad \text{yes, continuous!} \]

**Differentiable at $x=1$?**

\[ \lim_{x \to 1^-} g'(x) = \lim_{x \to 1^+} g'(x) ? \]

\[ \lim_{x \to 1^-} 1 = \lim_{x \to 1^+} 6x^5 ? \]

\[ 1 = 6? \]

\[ 1 = 6 \quad \text{NO!} \]

**Thm 2.1**

Differentiability $\not\Rightarrow$ Continuity

But Continuity $\not\Rightarrow$ Differentiability
We already discovered that if \( f(x) = x^n \), then \( f'(x) = nx^{n-1} \).

**Power Rule**: \[
\frac{d}{dx} \left[ x^n \right] = nx^{n-1}
\]
THM 3.1
\[
\frac{d}{dx} \left[ k \cdot b(x) \right] = k \frac{d}{dx} \left[ b(x) \right]
\]

\[ \text{Ex.} \quad \text{if} \quad b(x) = 7x^6, \text{ then} \]
\[
b'(x) = 7 \frac{d}{dx} (x^6) = 7 \cdot (6x^5) = 42x^5
\]

THM 3.2
\[
\frac{d}{dx} \left[ b(x) \pm g(x) \right] = b'(x) \pm g'(x)
\]

\[ \text{Ex.} \quad \text{Find} \quad g'(x) \quad \text{if} \quad g(x) = x^9 + x^2 - 5x + \frac{1}{2} \]
\[
g'(x) = \frac{d}{dx} \left[ x^9 \right] + \frac{d}{dx} \left[ x^2 \right] - \frac{d}{dx} \left[ 5x \right] + \frac{d}{dx} \left[ \frac{1}{2} \right]
\]
\[
= 9x^8 + 2x - 5 + 0
\]
\[
= 9x^8 + 2x - 5
\]
Find the derivative of the following:

1. \( g(t) = \sqrt{t} - \pi t^3 = t^{\frac{1}{2}} - \pi t^3 \)

2. \( v(t) = \frac{2t^3 - t}{t^2} = \frac{2t^3}{t^2} - \frac{t}{t^2} = 2t - \frac{1}{t} \)

3. \( y = \sqrt[5]{x} - \frac{7}{x} + \frac{11}{x^2} = x^{\frac{1}{5}} - 7x^{-1} + 11x^{-2} \)

4. \( f(x) = e^x e^{-\pi} - \pi x^\pi + 411 \)

5. \( g'(t) = \frac{1}{2} t^{-\frac{1}{2}} - \pi(3t^2) \)

6. \( v'(t) = 2 - (-t^{-2}) = 2 + \frac{1}{t^2} \)

7. \( \frac{dy}{dx} = \frac{4}{5} x^{-\frac{1}{5}} - 7(-x^{-2}) + 11(-2x^{-3}) \)

\[ = \frac{4}{5\sqrt[5]x} + \frac{7}{x^2} - \frac{22}{x^3} \]
\[ f(x) = e^x e^{-\pi x^2} + 411 \]

So:
\[ f'(x) = e \frac{d}{dx} \left[ e^x \right] - \pi e \frac{d}{dx} \left[ x^2 \right] + \frac{d}{dx} \left[ 411 \right] \]
\[ = e \cdot x - \pi \cdot 2x \]

\[ = e \cdot x - 2 \pi \cdot x \]

3.2 Exponential Functions

We'll investigate:
\[ y = 2^x \]
\[ y = \left( \frac{1}{3} \right)^x \]
\[ y = e^x \]
\[ y = \left( \frac{1}{3} \right)^x \]
\[ y_3 = \frac{y'}{y} \]

\[ \frac{y'}{y} = 2 - 1.0986 \]

\[ y_3 = 1.098612289 \]

\[ \ln(1/3) = -1.098612289 \]
\[ \ln(2) = 0.6931471806 \]
\[ \ln(e^x) = 1 \]

\[ \frac{d}{dx}[a^x] = \ln(a)a^x \]

\[ \frac{d}{dx}[e^x] = e^x \]